[1]Mass formulae for self-orthogonal, self-dual and LCD codes over finite commutative chain rings

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$\mathcal{R}_e$	a finite commutative chain ring with the nilpotency index e
u	a generator of the maximal ideal of $\mathcal{R}_e$
$\overline{\mathcal{R}}_e$	$\mathcal{R}_e/\langle u\rangle$, the residue field of $\mathcal{R}_e$

n positive integer

$\mathcal{R}_e^n$ $\mathcal{R}_e$-module consisting of all n-tuples over $\mathcal{R}_e$

Linear code

A linear code $\mathcal{C}$ of length n over $\mathcal{R}_e$ is defined as an $\mathcal{R}_e$-submodule of $\mathcal{R}_e^n$.

Generator matrix for a linear code

A generator matrix for a linear code $\mathcal{C}$ is defined as a matrix over $\mathcal{R}_e$ whose rows form a minimal generating set of the code $\mathcal{C}$.

Next for positive integers k and ℓ, let $M_{k\times\ell}(\mathcal{R}_e)$ denote the set of all $k\times\ell$ matrices over $\mathcal{R}_e$.

Theorem [Norton and Sălăgean (2000)]

Every linear code $\mathcal{C}$ of length n over $\mathcal{R}_e$ is permutation equivalent to a code with a generator matrix G in the standard form

$$G=\begin{bmatrix} I_{k_1} & A_{1,1} & A_{1,2} & \cdots & A_{1,e-1} & A_{1,e} \\ 0 & uI_{k_2} & uA_{2,2} & \cdots & uA_{2,e-1} & uA_{2,e} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & u^{e-2}A_{e-1,e-1} & u^{e-2}A_{e-1,e} \\ 0 & 0 & 0 & \cdots & u^{e-1}I_{k_e} & u^{e-1}A_{e,e} \end{bmatrix}, \qquad (1)$$

where the columns of the matrix G are grouped into blocks of sizes k_1, k_2, $\cdots$, k_{e-1}, k_e, $k_{e+1}=n-(k_1+k_2+\cdots+k_e)$, the matrix I_{k_i} is the $k_i\times\ k_i$ identity matrix over $\mathcal{R}_e$ and the matrix $A_{i,j}\in M_{k_i\times k_{j+1}}(\mathcal{R}_e)$ is considered modulo u^{j-i+1} for $1\le i\le j\le e$.

A linear code $\mathcal{C}$ of length n over $\mathcal{R}_e$ is said to be of the type $\{k_1,k_2,k_3,\cdots,k_e\}$ if it is permutation equivalent to a code with a generator matrix G of the form (1).

Euclidean bilinear form

The Euclidean bilinear form is a mapping $\langle \cdot, \cdot \rangle : \mathcal{R}_e^n \times \mathcal{R}_e^n \to \mathcal{R}_e$, defined as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

for $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathcal{R}_e^n$.

The dual code

The dual code $\mathcal{C}^\perp$ of a linear code $\mathcal{C}$ of length n over $\mathcal{R}_e$ is defined as

$$\mathcal{C}^\perp = \{y \in \mathcal{R}_e^n \mid \langle x, y \rangle = 0 \text{ for all } x \in \mathcal{C}\}.$$

Note that

1. the dual code $\mathcal{C}^\perp$ is also a linear code of length n over $\mathcal{R}_e$.
2. if the code $\mathcal{C}$ is of the type $\{k_1, k_2, \cdots, k_{e-1}, k_e\}$, then the dual code $\mathcal{C}^\perp$ is of the type $\{n - (k_1 + k_2 + \cdots + k_e), k_e, k_{e-1}, \cdots, k_2\}$.

Definition

A linear code $\mathcal{C}$ of length n over $\mathcal{R}_e$ is said to be

1. self-orthogonal if it satisfies $\mathcal{C} \subseteq \mathcal{C}^{\perp}$.
2. self-dual if it satisfies $\mathcal{C} = \mathcal{C}^{\perp}$.
3. linear with complementary dual (LCD) if it satisfies $\mathcal{C} \cap \mathcal{C}^{\perp} = \{0\}$.

For an integer k satisfying $0 \leq k \leq n$ and a prime power q, let

$\sigma_q(n,k)$ the number of distinct (Euclidean) self-orthogonal codes of length n and dimension k over the finite field $\mathbb{F}_q$

Note that

- $\sigma_q(n,0) = 1.$
- $\sigma_q(n,k) = 0$ for $k > \frac{n}{2}$.

Theorem [Pless (1968)]

For an integer k satisfying $1 \leq k \leq \frac{n}{2}$ and a prime power q, we have

$$\sigma_q(n,k) = \begin{cases} \dfrac{\prod_{i=0}^{k-1}(q^{n-1-2i}-1)}{\prod_{j=1}^{k}(q^j-1)} & \text{if } n \text{ is odd;} \\[2ex] \dfrac{(q^{n-k}-1)\prod_{i=1}^{k-1}(q^{n-2i}-1)}{\prod_{j=1}^{k}(q^j-1)} & \text{if both } n \text{ and } q \text{ are even;} \\[2ex] \dfrac{(q^{n-k}-q^{\frac{n}{2}-k}+q^{\frac{n}{2}}-1)\prod_{i=1}^{k-1}(q^{n-2i}-1)}{\prod_{j=1}^{k}(q^j-1)} & \text{if } n \text{ is even, } q \text{ is odd and } (-1)^{\frac{n}{2}} \text{ is a square in } \mathbb{F}_q; \\[2ex] \dfrac{(q^{n-k}+q^{\frac{n}{2}-k}-q^{\frac{n}{2}}-1)\prod_{i=1}^{k-1}(q^{n-2i}-1)}{\prod_{j=1}^{k}(q^j-1)} & \text{if } n \text{ is even, } q \text{ is odd and } (-1)^{\frac{n}{2}} \text{ is not a square in } \mathbb{F}_q. \end{cases}$$

From now on, let

$$
\begin{array}{rcl}
k_1, k_2, \cdots, k_{e+1} & & \text{non-negative integers, not all zero} \\
n & = & k_1 + k_2 + \cdots + k_{e+1}
\end{array}
$$

For integers t, ℓ satisfying $2 \leq t \leq \lceil \frac{e+1}{2} \rceil$ and $1 \leq \ell \leq t-1$, let us define

$$
\begin{aligned}
h_\ell(k_1, k_2, \cdots, k_t) &= (k_1 + k_2 + \cdots + k_\ell)\Big(n - (k_1 + k_2 + \cdots, k_{\ell+1}) - 1\Big), \\
n_\ell(k_1, k_2, \cdots, k_t) &= (k_1 + k_2 + \cdots + k_\ell)\Big(n - (k_1 + k_2 + \cdots, k_{\ell+1}) - 1\Big) \\
&\quad + \Big((k_1 + k_2 + \cdots + k_{t-\beta}) + (k_1 + k_2 + \cdots + k_t) \\
&\quad - (k_1 + k_2 + \cdots + k_{\ell+1})\Big)\Big(n - (k_1 + k_2 + \cdots + k_{t-\beta}) \\
&\quad - (k_1 + k_2 + \cdots + k_t)\Big),
\end{aligned}
$$

where $\beta = 1$ if e is even, while $\beta = 0$ if e is odd.

Mass formula for self-orthogonal codes of the type $\{k_1, k_2, \cdots, k_e\}$ and length n over $\mathcal{R}_e$ I

Theorem [M. Yadav & A. ____ (2021)]

Let $\mathcal{N}_e(n; k_1, k_2, \cdots, k_e)$ denote the number of distinct self-orthogonal codes of the type $\{k_1, k_2, \cdots, k_e\}$ and length n over $\mathcal{R}_e$. Let $\overline{\mathcal{R}}_e \simeq \mathbb{F}_{p^r}$, where p is an odd prime and r is a positive integer.

- When e is odd, we have

$$\mathcal{N}_e(n; k_1, k_2, \cdots, k_{e-1}, k_e)$$

$$= \begin{cases} \sigma_{p^r}\left(n, k_1 + k_2 + \cdots + k_{\frac{e+1}{2}}\right) \displaystyle\prod_{i=1}^{\frac{e+1}{2}} {k_1+k_2+\cdots+k_i \brack k_i}_{p^r} \\ \times \displaystyle\prod_{j=2}^{\frac{e+1}{2}} {k_j+k_{e+1}-k_1 \brack k_j}_{p^r} (p^r)^{\sum_{\ell=1}^{\frac{e-1}{2}} n_\ell(k_1, k_2, \cdots, k_{\frac{e+1}{2}})} \\ \quad \text{if } k_1 \leq k_{e+1} \text{ and } k_s = k_{e-s+2} \text{ for } 2 \leq s \leq e; \\ 0 \qquad \text{otherwise.} \end{cases}$$

Mass formula for self-orthogonal codes of the type $\{k_1, k_2, \cdots, k_e\}$ and length n over $\mathcal{R}_e$ II

- When e is even, we have

$$\mathcal{N}_e(n; k_1, k_2, \cdots, k_{e-1}, k_e)$$

$$= \begin{cases} \sigma_{p^r}\left(n, k_1 + k_2 + \cdots + k_{\frac{e}{2}}\right) \prod_{i=1}^{\frac{e}{2}} \begin{bmatrix} k_1+k_2+\cdots+k_i \\ k_i \end{bmatrix}_{p^r} \\ \times \prod_{j=2}^{\frac{e}{2}+1} \begin{bmatrix} k_j+k_{e+1}-k_1 \\ k_j \end{bmatrix}_{p^r} (p^r)^{\sum_{\ell=1}^{\frac{e}{2}} n_\ell(k_1,k_2,\cdots,k_{\frac{e}{2}+1}) + \Theta_e^\star(k_1,k_2,\cdots,k_{\frac{e}{2}+1})} \\ \quad \text{if } k_1 \leq k_{e+1} \text{ and } k_s = k_{e-s+2} \text{ for } 2 \leq s \leq e; \\ 0 \qquad \text{otherwise,} \end{cases}$$

where

$$\Theta_e^\star(k_1, k_2, \cdots, k_{\frac{e}{2}+1}) = -(k_1 + k_2 + \cdots + k_{\frac{e}{2}}) \left(\frac{k_1+k_2+\cdots+k_{\frac{e}{2}}+2k_{e+1}-2k_1-1}{2}\right).$$

Mass formula for self-orthogonal codes of length n over $\mathcal{R}_e$ I

Theorem [M. Yadav & A. ____ (2021)]

Let $\mathcal{N}_e(n)$ denote the number of distinct self-orthogonal codes of length n over $\mathcal{R}_e$. Let $\overline{\mathcal{R}_e} \simeq \mathbb{F}_{p^r}$, where p is an odd prime and r is a positive integer.

- When e is odd, we have

$$\mathcal{N}_e(n) = \sum_{\substack{k_1,k_2,\cdots,k_{\frac{e+1}{2}} \geq 0 \in \mathbb{N}\cup\{0\} \\ 0 \leq k_1+k_2+\cdots+k_{\frac{e+1}{2}} \leq \lfloor \frac{n}{2} \rfloor}} \sigma_{p^r}\left(n, k_1+k_2+\cdots+k_{\frac{e+1}{2}}\right)$$

$$\times \prod_{i=1}^{\frac{e+1}{2}} \begin{bmatrix} k_1+k_2+\cdots+k_i \\ k_i \end{bmatrix}_{p^r} (p^r)^{\sum_{\ell=1}^{\frac{e-1}{2}} n_\ell(k_1,k_2,\cdots,k_{\frac{e+1}{2}})}$$

$$\times \prod_{j=2}^{\frac{e+1}{2}} \begin{bmatrix} k_j+n-2(k_1+k_2+\cdots+k_{\frac{e+1}{2}}) \\ k_j \end{bmatrix}_{p^r}.$$

Mass formula for self-orthogonal codes of length n over $\mathcal{R}_e$ II

- When e is even, we have

$$\mathcal{N}_e(n) = \sum_{\substack{k_1,k_2,\cdots,k_{\frac{e}{2}+1}\in\mathbb{N}\cup\{0\} \\ 0\le 2k_1+\cdots+2k_{\frac{e}{2}}+k_{\frac{e}{2}+1}\le n}} \sigma_{p^r}\left(n, k_1+k_2+\cdots+k_{\frac{e}{2}}\right)$$

$$\times \prod_{i=1}^{\frac{e}{2}} \begin{bmatrix} k_1+k_2+\cdots+k_i \\ k_i \end{bmatrix}_{p^r} (p^r)^{\sum_{\ell=1}^{\frac{e}{2}} n_\ell(k_1,k_2,\cdots,k_{\frac{e}{2}+1})+\Theta_e(k_1,k_2,\cdots,k_{\frac{e}{2}+1})}$$

$$\times \prod_{j=2}^{\frac{e}{2}+1} \begin{bmatrix} k_j+n-2(k_1+k_2+\cdots+k_{\frac{e}{2}})-k_{\frac{e}{2}+1} \\ k_j \end{bmatrix}_{p^r},$$

where

$$\Theta_e(k_1,k_2,\cdots,k_{\frac{e}{2}+1}) = -(k_1+k_2+\cdots+k_{\frac{e}{2}})\left(\frac{2n-3(k_1+k_2+\cdots+k_{\frac{e}{2}})-2k_{\frac{e}{2}+1}-1}{2}\right).$$

(Here $\mathbb{N}$ denotes the set of positive integers.)

Mass formula for self-dual codes of the type $\{k_1, k_2, \cdots, k_e\}$ and length n over $\mathcal{R}_e$ I

Theorem [M. Yadav & A. ____ (2021)]

Let $\mathcal{M}_e(n; k_1, k_2, \cdots, k_e)$ denote the number of distinct self-dual codes of the type $\{k_1, k_2, \cdots, k_e\}$ and length n over $\mathcal{R}_e$. Let $\overline{\mathcal{R}_e} \simeq \mathbb{F}_{p^r}$, where p is an odd prime and r is a positive integer.

- When e is even, we have

$$\mathcal{M}_e(n; k_1, k_2, \cdots, k_e)$$

$$= \begin{cases} \sigma_{p^r}\left(n, k_1 + k_2 + \cdots + k_{\frac{e}{2}}\right) \prod\limits_{i=1}^{\frac{e}{2}} {k_1+k_2+\cdots+k_i \brack k_i}_{p^r} \\ \times (p^r)^{\sum\limits_{\ell=1}^{\frac{e}{2}} h_\ell(k_1,k_2,\cdots,k_{\frac{e}{2}+1}) + \lambda_e(k_1,k_2,\cdots,k_{\frac{e}{2}+1})} \\ \text{if } k_s = k_{e-s+2} \ \text{ for } \ 1 \leq s \leq e+1; \\ \\ 0 \qquad \text{otherwise,} \end{cases}$$

where $\lambda_e(k_1, k_2, \cdots, k_{\frac{e}{2}+1}) = -(k_1 + k_2 + \cdots + k_{\frac{e}{2}}) \left(\dfrac{k_1+k_2+\cdots+k_{\frac{e}{2}}-1}{2}\right).$

Mass formula for self-dual codes of the type $\{k_1, k_2, \cdots, k_e\}$ and length n over $\mathcal{R}_e$ II

- When e is odd, we have

$$\mathcal{M}_e(n; k_1, k_2, \cdots, k_e)$$

$$= \begin{cases} 2 \prod_{b=1}^{\frac{n}{2}-1} (p^{rb}+1) \prod_{i=1}^{\frac{e+1}{2}} \begin{bmatrix} k_1+k_2+\cdots+k_i \\ k_i \end{bmatrix}_{p^r} (p^r)^{\sum_{\ell=1}^{\frac{e-1}{2}} h_\ell(k_1, k_2, \cdots, k_{\frac{e+1}{2}})} \\ \text{if } n \text{ is even, } (-1)^{\frac{n}{2}} \text{ is a square in } \overline{\mathcal{R}}_e \text{ and } k_s = k_{e-s+2} \text{ for } 1 \le s \le e+1; \\ 0 \qquad \text{otherwise.} \end{cases}$$

Mass formula for self-dual codes of length n over $\mathcal{R}_e$ I

Theorem [M. Yadav & A. ____ (2021)]

Let $\mathcal{M}_e(n)$ denote the number of distinct self-dual codes of length n over $\mathcal{R}_e$. Let $\overline{\mathcal{R}_e} \simeq \mathbb{F}_{p^r}$, where p is an odd prime and r is a positive integer.

- When e is even, we have

$$\mathcal{M}_e(n) = \sum_{\substack{k_1,k_2,\cdots,k_{\frac{e}{2}} \in \mathbb{N}\cup\{0\} \\ 0 \leq k_1+k_2+\cdots+k_{\frac{e}{2}} \leq \lfloor \frac{n}{2} \rfloor}} \sigma_{p^r}(n, k_1+k_2+\cdots+k_{\frac{e}{2}}) \prod_{i=1}^{\frac{e}{2}} \begin{bmatrix} k_1+k_2+\cdots+k_i \\ k_i \end{bmatrix}_{p^r}$$

$$\times (p^r)^{\sum_{\ell=1}^{\frac{e}{2}-1} h_\ell(k_1,k_2,\cdots,k_{\frac{e}{2}}) + \lambda'_e(k_1,k_2,\cdots,k_{\frac{e}{2}})},$$

where $\lambda'_e(k_1,k_2,\cdots,k_{\frac{e}{2}}) = (k_1+k_2+\cdots+k_{\frac{e}{2}}) \left(\frac{k_1+k_2+\cdots+k_{\frac{e}{2}}-1}{2} \right)$.

Mass formula for self-dual codes of length n over $\mathcal{R}_e$ II

- When e is odd, we have

$$\mathcal{M}_e(n) = \begin{cases} \sum\limits_{\substack{k_1,k_2,\cdots,k_{\frac{e-1}{2}} \in \mathbb{N}\cup\{0\} \\ 0\le k_1+k_2+\cdots+k_{\frac{e-1}{2}} \le \frac{n}{2}}} 2\prod\limits_{b=1}^{\frac{n}{2}-1}(p^{rb}+1)\prod\limits_{i=1}^{\frac{e-1}{2}} \begin{bmatrix} k_1+k_2+\cdots+k_i \\ k_i \end{bmatrix}_{p^r} \\ \times \begin{bmatrix} \frac{n}{2} \\ k_1+k_2+\cdots+k_{\frac{e-1}{2}} \end{bmatrix}_{p^r} (p^r)^{\sum_{\ell=1}^{\frac{e-3}{2}} h_\ell(k_1,k_2,\cdots,k_{\frac{e-1}{2}})+\lambda_e^*(k_1,k_2,\cdots,k_{\frac{e-1}{2}})} \\ \quad \text{if } n \text{ is even and } (-1)^{\frac{n}{2}} \text{ is a square in } \overline{\mathcal{R}}_e; \\ 0 \qquad \text{otherwise,} \end{cases}$$

where $\lambda_e^*(k_1,k_2,\cdots,k_{\frac{e-1}{2}}) = \left(\frac{n}{2}-1\right)\left(k_1+k_2+\cdots+k_{\frac{e-1}{2}}\right)$.

LCD codes over $\mathcal{R}_e$

Theorem [Bhowmick *et al.* (2020)]

Any LCD code C of length n over the finite commutative Frobenius ring R is a free code, i.e., the code C is a free R-submodule of R^n.

Any LCD code $\mathcal{C}$ of length n over the finite commutative chain ring $\mathcal{R}_e$ is a free code, i.e., the code $\mathcal{C}$ is a free $\mathcal{R}_e$-submodule of $\mathcal{R}_e^n$.

As a consequence, the LCD code $\mathcal{C}$ is permutation equivalent to a code whose generator matrix G is in the standard form

$$G = [I_k \mid A],$$

where I_k is the $k \times k$ identity matrix and A is a $k \times (n-k)$ matrix over $\mathcal{R}_e$.

The integer k is called the rank of the code $\mathcal{C}$.

Mass formula for LCD codes of length n and rank k over $\mathcal{R}_e$ I

Theorem [M. Yadav & A. ____ (2021)]

For $0 \leq k \leq n$, let $\mathcal{L}_e(n;k)$ denote the number of distinct LCD codes of length n and rank k over $\mathcal{R}_e$. Let $\overline{\mathcal{R}}_e \simeq \mathbb{F}_{p^r}$, where p is a prime and r is a positive integer. Then we have $\mathcal{L}_e(n;0) = \mathcal{L}_e(n;n) = 1$. Further, for $1 \leq k \leq n-1$, we have the following:

- When $p = 2$, we have

$$\mathcal{L}_e(n;k) = \begin{cases} 2^{\frac{r(n-k)(2k\ell-k+1)}{2}} \begin{bmatrix} (n-1)/2 \\ (k-1)/2 \end{bmatrix}_{2^{2r}} & \text{if both } k \text{ and } n \text{ are odd;} \\ 2^{\frac{r(k(n-k)(2\ell-1)+n-1)}{2}} \begin{bmatrix} (n-2)/2 \\ (k-1)/2 \end{bmatrix}_{2^{2r}} & \text{if } k \text{ is odd and } n \text{ is even;} \\ 2^{\frac{rk((n-k)(2\ell-1)+1)}{2}} \begin{bmatrix} (n-1)/2 \\ k/2 \end{bmatrix}_{2^{2r}} & \text{if } k \text{ is even and } n \text{ is odd;} \\ 2^{\frac{r(k(n-k)(2\ell-1)-2)}{2}} \Big((2^{rk} + 2^r - 1) \begin{bmatrix} (n-2)/2 \\ k/2 \end{bmatrix}_{2^{2r}} & \\ +(2^{r(n-k+1)} - 2^{r(n-k)} + 1) \begin{bmatrix} (n-2)/2 \\ (k-2)/2 \end{bmatrix}_{2^{2r}} \Big) & \text{if both } k \text{ and } n \text{ are even.} \end{cases}$$

Mass formula for LCD codes of length n and rank k over $\mathcal{R}_e$ II

- When p is an odd prime, we have

$$\mathcal{L}_e(n;k)=\begin{cases} p^{\frac{r(n-k)(2k\ell-k+1)}{2}}\begin{bmatrix}(n-1)/2\\(k-1)/2\end{bmatrix}_{p^{2r}} & \text{if both } k \text{ and } n \text{ are odd;}\\ p^{\frac{r(k(n-k)(2\ell-1)-1)}{2}}(p^{\frac{rn}{2}}-1)\begin{bmatrix}(n-2)/2\\(k-1)/2\end{bmatrix}_{p^{2r}} & \text{if } k \text{ is odd and } n \text{ is even}\\ \text{with either } p^r\equiv 1 \pmod 4 \text{ or } n\equiv 0 \pmod 4 \text{ and } p^r\equiv 3 \pmod 4; & \\ p^{\frac{r(k(n-k)(2\ell-1)-1)}{2}}(p^{\frac{rn}{2}}+1)\begin{bmatrix}(n-2)/2\\(k-1)/2\end{bmatrix}_{p^{2r}} & \text{if } k \text{ is odd, } n \text{ is even,}\\ p^r\equiv 3 \pmod 4 \text{ and } n\equiv 2 \pmod 4; & \\ p^{\frac{rk((n-k)(2\ell-1)+1)}{2}}\begin{bmatrix}(n-1)/2\\k/2\end{bmatrix}_{p^{2r}} & \text{if } k \text{ is even and } n \text{ is odd;}\\ p^{\frac{rk(n-k)(2\ell-1)}{2}}\begin{bmatrix}n/2\\k/2\end{bmatrix}_{p^{2r}} & \text{if both } k \text{ and } n \text{ are even.} \end{cases}$$

Mass formula for LCD codes of length n over $\mathcal{R}_e$ I

Theorem [M. Yadav & A. ____ (2021)]

Let $\mathcal{L}_e(n)$ denote the number of distinct LCD codes of length n over $\mathcal{R}_e$. Let $\overline{\mathcal{R}}_e \simeq \mathbb{F}_{p^r}$, where p is a prime and r is a positive integer.

- When $p = 2$, we have

$$\mathcal{L}_e(n) = \begin{cases} 2 + \displaystyle\sum_{\substack{k=1 \\ k\equiv 0 \pmod 2}}^{n-1} 2^{\frac{rk((n-k)(2\ell-1)+1)}{2}} {\begin{bmatrix} (n-1)/2 \\ k/2 \end{bmatrix}}_{2^{2r}} \\ + \displaystyle\sum_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{n-1} 2^{\frac{r(n-k)(2k\ell-k+1)}{2}} {\begin{bmatrix} (n-1)/2 \\ (k-1)/2 \end{bmatrix}}_{2^{2r}} & \text{if } n \text{ is odd;} \\ \\ 2 + \displaystyle\sum_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{n-1} 2^{\frac{r(k(n-k)(2\ell-1)+n-1)}{2}} {\begin{bmatrix} (n-2)/2 \\ (k-1)/2 \end{bmatrix}}_{2^{2r}} \\ + \displaystyle\sum_{\substack{k=1 \\ k\equiv 0 \pmod 2}}^{n-1} 2^{\frac{r(k(n-k)(2\ell-1)-2)}{2}} \Big((2^{rk} + 2^r - 1) {\begin{bmatrix} (n-2)/2 \\ k/2 \end{bmatrix}}_{2^{2r}} \\ + (2^{r(n-k+1)} - 2^{r(n-k)} + 1) {\begin{bmatrix} (n-2)/2 \\ (k-2)/2 \end{bmatrix}}_{2^{2r}} \Big) & \text{if } n \text{ is even.} \end{cases}$$

Mass formula for LCD codes of length n over $\mathcal{R}_e$ II

- When p is an odd prime and n is even, we have

$$\mathcal{L}_e(n) = \begin{cases} 2 + \sum\limits_{\substack{k=1 \\ k\equiv 0 \pmod 2}}^{n-1} p^{\frac{rk(n-k)(2\ell-1)}{2}} \begin{bmatrix} n/2 \\ k/2 \end{bmatrix}_{p^{2r}} \\ + \sum\limits_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{n-1} p^{\frac{r(k(n-k)(2\ell-1)-1)}{2}} (p^{\frac{rn}{2}} - 1) \begin{bmatrix} (n-2)/2 \\ (k-1)/2 \end{bmatrix}_{p^{2r}} \\ \text{if either } p^r \equiv 1 \pmod 4 \text{ or } n \equiv 0 \pmod 4 \text{ and } p^r \equiv 3 \pmod 4; \\ \\ 2 + \sum\limits_{\substack{k=1 \\ k\equiv 0 \pmod 2}}^{n-1} p^{\frac{rk(n-k)(2\ell-1)}{2}} \begin{bmatrix} n/2 \\ k/2 \end{bmatrix}_{p^{2r}} \\ + \sum\limits_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{n-1} p^{\frac{r(k(n-k)(2\ell-1)-1)}{2}} (p^{\frac{rn}{2}} + 1) \begin{bmatrix} (n-2)/2 \\ (k-1)/2 \end{bmatrix}_{p^{2r}} \\ \text{if } p^r \equiv 3 \pmod 4 \text{ and } n \equiv 2 \pmod 4. \end{cases}$$

Mass formula for LCD codes of length n over $\mathcal{R}_e$ III

- When p is an odd prime and n is odd, we have

$$\begin{aligned}\mathcal{L}_e(n) &= 2 + \sum_{\substack{k=1 \\ k\equiv 1 \pmod 2}}^{n-1} p^{\frac{r(n-k)(2k\ell-k+1)}{2}} \begin{bmatrix} (n-1)/2 \\ (k-1)/2 \end{bmatrix}_{p^{2r}} \\ &\quad + \sum_{\substack{k=1 \\ k\equiv 0 \pmod 2}}^{n-1} p^{\frac{rk((n-k)(2\ell-1)+1)}{2}} \begin{bmatrix} (n-1)/2 \\ k/2 \end{bmatrix}_{p^{2r}}.\end{aligned}$$

Classification of self-orthogonal, self-dual and LCD codes over $\mathcal{R}_e$

Two self-orthogonal (resp. self-dual, LCD) codes of length n over $\mathcal{R}_e$ are said to be equivalent if one code can be obtained from the other by a combination of operations of the following two types:

A. Permutation of the n coordinate positions of the code.

B. Multiplication of the code symbols appearing in a given coordinate position by the element $-1 \in \mathcal{R}_e$.

Total number of self-orthogonal and inequivalent self-orthogonal codes of a given type and length 3 over $\mathbb{F}_5[u]/<u^2>$

Type $\{k_1, k_2\}$	Total number of self-orthogonal codes of the type $\{k_1, k_2\}$ and length 3 over $\mathbb{F}_5[u]/<u^2>$	Number of inequivalent self-orthogonal codes of the type $\{k_1, k_2\}$ and length 3 over $\mathbb{F}_5[u]/<u^2>$
$\{1,0\}$	30	2
$\{1,1\}$	6	1
$\{0,1\}$	31	5
$\{0,2\}$	31	5
$\{0,3\}$	1	1

There are precisely 14 inequivalent non-zero self-orthogonal codes of length 3 over $\mathbb{F}_5[u]/<u^2>$ with generator matrices

$$uI_3, \quad \begin{bmatrix}1 & 0 & 2\end{bmatrix}, \quad \begin{bmatrix}1 & u & 2\end{bmatrix}, \quad \begin{bmatrix}u & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}u & u & u\end{bmatrix},$$

$$\begin{bmatrix}u & 2u & u\end{bmatrix}, \quad \begin{bmatrix}u & u & 0\end{bmatrix}, \quad \begin{bmatrix}u & 0 & 3u\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 2\\ 0 & u & 0\end{bmatrix}, \quad \begin{bmatrix}u & 0 & 0\\ 0 & u & 0\end{bmatrix},$$

$$\begin{bmatrix}u & 0 & u\\ 0 & u & 0\end{bmatrix}, \quad \begin{bmatrix}u & 0 & 2u\\ 0 & u & 0\end{bmatrix}, \quad \begin{bmatrix}u & 0 & 4u\\ 0 & u & 2u\end{bmatrix}, \quad \begin{bmatrix}u & 0 & 4u\\ 0 & u & u\end{bmatrix}.$$

Total number of self-orthogonal and inequivalent self-orthogonal codes of a given type and length 4 over $\mathbb{F}_5[u]/<u^2>$

Type $\{k_1, k_2\}$	Total number of self-orthogonal codes of the type $\{k_1, k_2\}$ and length 4 over $\mathbb{F}_5[u]/<u^2>$	Number of inequivalent self-orthogonal codes of the type $\{k_1, k_2\}$ and length 4 over $\mathbb{F}_5[u]/<u^2>$
$\{1,0\}$	900	10
$\{1,1\}$	1080	14
$\{1,2\}$	36	2
$\{2,0\}$	60	2
$\{0,1\}$	156	8
$\{0,2\}$	806	18
$\{0,3\}$	156	8
$\{0,4\}$	1	1

There are precisely 63 inequivalent non-zero self-orthogonal codes of length 4 over $\mathbb{F}_5[u]/ < u^2 >$, whose generator matrices are as listed below:

- $\begin{bmatrix} 1 & xu & yu & 2 \end{bmatrix}$ with $(x, y) \in \{(0,0),(1,1),(1,2),(0,2)\}$;
- $\begin{bmatrix} 1 & xu+1 & yu+2 & zu+2 \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0),(0,1,4),(1,0,2),(1,3,4),(0,3,2),(2,4,0)\}$;
- $\begin{bmatrix} 1 & 0 & xu & 2 \\ 0 & u & yu & 0 \end{bmatrix}$ with $(x, y) \in \{(0,0),(1,0),(0,1),(1,1),(0,2),(1,2)\}$;
- $\begin{bmatrix} 1 & 1 & xu+2 & yu+2 \\ 0 & u & zu & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0,0,0,2),(1,4,0,2),(2,3,0,2),(0,0,1,1),(1,4,1,1),(3,2,1,1),(0,0,3,4),(1,4,3,4)\}$;
- uI_4, $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & u & 2 \\ 0 & 1 & 2 & 4u \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & u & 0 & 2u \\ 0 & 0 & u & 4u \end{bmatrix}$;
- $\begin{bmatrix} u & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0),(1,0,0),(1,1,0),(1,1,1),(2,3,4),(1,2,4),(2,1,0),(2,0,0)\}$;
- $\begin{bmatrix} u & 0 & xu & yu \\ 0 & u & zu & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0,0,0,0),(1,0,0,0),(1,1,0,0),(3,0,2,0),(0,3,1,2),(1,0,0,1),(4,0,2,1),(3,4,2,1),(1,0,0,2),(0,0,0,2),(0,0,1,2),(4,2,1,2),(4,2,2,2),(4,2,2,1),(4,4,4,0),(4,2,4,1),(2,0,0,2),(1,0,1,0)\}$;
- $\begin{bmatrix} u & 0 & 0 & xu \\ 0 & u & 0 & yu \\ 0 & 0 & u & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0),(1,0,0),(4,1,0),(2,0,0),(3,1,0),(1,4,1),(4,4,3),(2,4,3)\}$.

Total number of self-orthogonal and inequivalent self-orthogonal codes of a given type and length 5 over $\mathbb{F}_5[u]/<u^2>$

Type $\{k_1, k_2\}$	Total number of self-orthogonal codes of the type $\{k_1, k_2\}$ and length 5 over $\mathbb{F}_5[u]/<u^2>$	Number of inequivalent self-orthogonal codes of the type $\{k_1, k_2\}$ and length 5 over $\mathbb{F}_5[u]/<u^2>$
$\{1,0\}$	19500	27
$\{1,1\}$	120900	109
$\{1,2\}$	24180	41
$\{1,3\}$	156	3
$\{0,1\}$	781	11
$\{0,2\}$	20306	49
$\{0,3\}$	20306	49
$\{0,4\}$	781	11
$\{0,5\}$	1	1
$\{2,0\}$	19500	16
$\{2,1\}$	780	4

There are precisely 321 inequivalent non-zero self-orthogonal codes of length 5 over $\mathbb{F}_5[u]/<u^2>$, whose generator matrices are as listed below:

- $\begin{bmatrix}1 & xu & yu & zu & 2\end{bmatrix}$ with $(x,y,z) \in \{(0,0,0),(0,0,1),(0,1,1),(0,1,2),(1,1,1),(1,1,2)\}$;
- $\begin{bmatrix}1 & xu & yu+1 & zu+2 & wu+2\end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(0,0,1,4),(0,0,2,3),(0,1,0,2),(0,1,3,4),(0,2,0,4),(3,0,2,3),(1,0,0,0),(1,0,1,4),(1,0,2,3),(1,1,0,2),(1,1,1,1),(1,1,3,4),(1,2,0,4),(1,2,1,3)\}$;
- $\begin{bmatrix}1 & xu+1 & yu+1 & zu+1 & wu+1\end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(0,0,1,4),(0,0,2,3),(0,1,1,3),(0,1,2,2),(1,2,3,4)\}$;
- $\begin{bmatrix}1 & 0 & xu & yu & 2\\ 0 & u & zu & wu & 0\end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(0,1,0,0),(1,1,0,0),(1,2,0,0),(4,2,0,1),(0,0,0,1),(0,1,0,1),(1,0,0,1),(1,1,0,1),(4,2,1,1),(0,0,1,1),(0,1,1,1),(0,0,0,2),(0,0,1,2),(0,1,0,2),(0,1,1,2),(0,1,2,0),(0,1,2,1),(1,1,4,3),(1,1,0,2),(1,1,1,3)\}$;
- $\begin{bmatrix}1 & 0 & 1 & 2 & 2\\ 0 & u & xu & yu & zu\end{bmatrix}$ with $(x,y,z) \in \{(0,0,0),(0,1,4),(0,2,3),(1,0,2),(1,3,4),(2,1,3)\}$;
- $\begin{bmatrix}1 & 0 & 1 & u+2 & 4u+2\\ 0 & u & xu & yu & zu\end{bmatrix}$ with $(x,y,z) \in \{(0,0,0),(0,1,4),(0,2,3),(1,0,2),(1,1,1),(1,3,4),(2,0,4),(2,1,3),(3,3,3)\}$;
- $\begin{bmatrix}1 & 0 & 1 & 2u+2 & 3u+2\\ 0 & u & xu & yu & zu\end{bmatrix}$ with $(x,y,z) \in \{(0,0,0),(0,1,4),(0,2,3),(1,0,2),(1,1,1),(1,3,4),(2,0,4),(2,1,3),(2,2,2)\}$;

- $\begin{bmatrix} 1 & 0 & u+1 & 2 & 2u+2 \\ 0 & u & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0), (0,1,4), (0,2,3), (1,0,2), (1,2,0), (1,3,4), (2,1,3)\}$;
- $\begin{bmatrix} 1 & 0 & u+1 & 3u+2 & 4u+2 \\ 0 & u & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0), (0,1,4), (0,2,3), (1,0,2), (1,1,1), (1,2,0), (1,4,3), (1,3,4), (2,0,4), (2,1,3), (2,2,2), (2,3,1), (2,4,0)\}$;
- $\begin{bmatrix} 1 & 0 & 2u+1 & 2 & 4u+2 \\ 0 & u & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0), (0,1,4), (0,2,3), (1,0,2), (1,2,0), (1,3,4), (2,1,3)\}$;
- $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & u & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,4), (0,1,3), (2,3,4)\}$;
- $\begin{bmatrix} 1 & 1 & 1 & u+1 & 4u+1 \\ 0 & u & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,4), (0,1,3), (0,2,2), (0,4,0), (2,0,2), (2,1,1), (2,3,4)\}$;
- $\begin{bmatrix} 1 & 1 & xu+1 & yu+1 & 3u+1 \\ 0 & u & zu & wu & su \end{bmatrix}$ with $(x, y, z, w, s) \in \{(0,2,0,0,4), (0,2,0,4,0), (0,2,2,0,2), (0,2,2,1,1), (1,1,0,2,2), (1,1,2,4,3)\}$;
- $\begin{bmatrix} 1 & 3 & 1 & 3 & xu \\ 0 & u & yu & zu & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(1,0,4,0), (1,1,2,0), (1,2,0,0), (1,4,1,0), (0,0,4,0), (0,1,2,0), (0,2,0,0)\}$;

- $\begin{bmatrix} 1 & 3 & u+1 & 3u+3 & xu \\ 0 & u & yu & zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,4,0),(0,1,2,0),$ $(0,2,0,0),(1,0,4,0),(1,1,2,0),(1,3,3,0),(1,4,1,0),(1,2,0,0)\}$;
- $\begin{bmatrix} 1 & 3 & 3u+1 & 4u+3 & xu \\ 0 & u & yu & zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,2,0,0),(1,2,0,0),$ $(1,0,4,0)\}$;
- uI_5, $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & u+2 & 3u+3 & u+4 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & u+1 & 2u+1 & 2u+1 \\ 0 & u & 3u & 2u & 4u \end{bmatrix}$;
- $\begin{bmatrix} 1 & 3 & 2u+1 & u+3 & xu \\ 0 & u & 4u & u & 0 \end{bmatrix}$ with $x \in \{0,1\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 & 2 & 2 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 2u \\ 0 & 0 & 0 & u & 4u \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & u & 0 & 0 & 4u \\ 0 & 0 & u & u & 3u \\ 0 & 0 & 0 & u & 4u \end{bmatrix}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & 2 \\ 0 & u & 0 & yu & 0 \\ 0 & 0 & u & zu & 0 \end{bmatrix}$ with $(x,y,z) \in \{(0,0,0),(0,0,1),(0,0,2),(1,0,0),(1,0,1),$ $(1,0,2),(3,1,1),(4,1,2),(0,1,3),(0,1,1)\}$;

- $\begin{bmatrix} 1 & 0 & 1 & xu+2 & yu+2 \\ 0 & u & 0 & zu & wu \\ 0 & 0 & u & vu & su \end{bmatrix}$ with $(x,y,z,w,v,s) \in \{(0,0,0,0,0,2),$ $(0,0,0,0,1,1),(0,0,0,0,3,4),(0,0,1,4,3,4),(0,0,1,4,0,2),(0,0,1,4,1,1),$ $(1,4,1,4,0,2),(0,0,2,3,1,1),(0,0,2,3,3,4),(1,4,0,0,2,0),(1,4,0,0,1,1),$ $(1,4,0,0,3,4),(1,4,1,4,4,3),(1,4,1,4,1,1),(1,4,2,3,3,4),(1,4,2,3,1,1),$ $(2,3,0,0,2,0),(2,3,0,0,1,1),(2,3,1,4,1,1),(2,3,1,4,0,2),(2,3,2,3,1,1)\}$;
- $\begin{bmatrix} 1 & 1 & 1 & xu+1 & yu+1 \\ 0 & u & 0 & zu & wu \\ 0 & 0 & u & vu & su \end{bmatrix}$ with $(x,y,z,w,v,s) \in \{(0,0,0,4,1,3),$ $(0,0,0,4,2,2),(1,4,0,4,4,0),(1,4,0,4,0,4),(0,0,1,3,3,1),(2,3,0,4,1,3),$ $(2,3,0,4,2,2),(1,4,1,3,3,1)\}$;
- $\begin{bmatrix} 1 & 3 & 1 & 3 & xu \\ 0 & u & 0 & 4u & 0 \\ 0 & 0 & u & 3u & 0 \end{bmatrix}$ with $x \in \{0,1\}$;
- $\begin{bmatrix} u & xu & yu & zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,2),(0,1,2,2),(1,1,1,1),$ $(0,0,0,0),(1,0,0,0),(1,1,0,0),(1,1,1,0),(2,1,1,0),(2,3,1,1),(2,3,3,3),(2,1,0,0)\}$;

- $\begin{bmatrix} u & 0 & xu & yu & zu \\ 0 & u & wu & vu & su \end{bmatrix}$ with $(x, y, z, w, v, s) \in \{(0,0,0,0,0,0),$
$(0,0,0,0,0,1), (0,0,0,0,0,2), (0,0,0,0,1,1), (0,0,0,0,1,2), (0,0,0,1,1,1),$
$(0,0,0,1,1,2), (0,0,0,1,3,3), (0,0,1,4,3,1), (0,0,1,0,0,1), (0,0,1,0,0,2),$
$(0,0,1,0,1,4), (0,0,1,0,1,0), (0,0,1,0,1,2), (0,0,1,0,2,0), (0,0,1,0,2,1),$
$(0,0,1,1,1,1), (0,0,1,1,1,2), (0,0,1,1,2,0), (0,0,1,1,2,2), (0,0,1,1,4,0),$
$(0,1,1,3,4,0), (0,1,1,3,4,1), (0,1,1,4,2,1), (0,1,1,1,1,3), (0,1,1,1,2,0),$
$(0,1,1,1,4,0), (0,1,1,2,0,0), (0,1,1,0,4,3), (0,1,1,0,4,2), (0,1,1,0,3,3),$
$(0,1,2,2,4,3), (0,1,2,0,2,0), (0,1,2,0,2,1), (0,1,2,1,0,2), (0,1,2,1,1,1),$
$(0,1,2,1,1,3), (0,1,2,1,2,0), (0,1,2,1,2,1), (0,1,2,1,2,2), (0,1,2,2,0,0),$
$(0,1,2,2,1,1), (0,1,2,2,1,3), (2,3,2,0,0,2), (1,2,3,1,1,1), (1,2,2,1,3,4),$
$(1,2,2,2,1,3), (1,1,1,0,0,2), (0,0,2,3,0,0)\};$

- $\begin{bmatrix} u & 0 & 0 & xu & yu \\ 0 & u & 0 & zu & wz \\ 0 & 0 & u & vu & su \end{bmatrix}$ with $(x, y, z, w, v, s) \in \{(0,0,0,0,0,0),$
$(0,0,0,0,0,1), (0,0,0,0,0,2), (0,0,0,0,1,1), (0,0,0,0,1,2), (0,0,0,1,4,4),$
$(0,0,0,1,0,1), (0,0,0,1,0,2), (0,0,0,1,1,0), (0,0,0,1,1,2), (0,0,0,1,2,0),$
$(0,0,0,2,1,4), (0,0,0,1,2,1), (0,0,0,2,2,3), (0,0,0,2,2,0), (0,0,1,1,1,2),$
$(0,0,1,1,1,3), (0,0,2,3,4,3), (0,1,2,0,4,4), (0,1,3,2,0,1), (0,1,3,3,0,2),$
$(0,1,3,0,0,4), (0,1,4,0,1,4), (0,1,4,2,1,1), (0,1,4,3,1,2), (0,1,1,4,3,3),$
$(0,1,1,3,3,2), (0,1,1,2,3,1), (0,1,0,1,4,4), (0,1,0,1,0,1), (0,1,0,1,0,2),$
$(0,1,0,1,1,3), (0,1,0,1,1,0), (0,1,0,2,4,3), (0,1,0,2,0,2), (0,1,0,2,1,4),$
$(0,1,0,2,1,0), (0,1,0,2,2,0), (0,1,1,4,4,4), (0,1,1,4,2,4), (0,1,1,0,2,2),$
$(1,1,4,3,4,1), (1,1,4,3,0,2), (1,1,4,3,2,4), (2,2,3,0,4,4), (2,2,1,4,0,2),$
$(2,2,4,2,1,3), (1,2,2,1,1,4), (1,2,2,1,2,3)\};$

- $\begin{bmatrix} u & 0 & 0 & 0 & xu \\ 0 & u & 0 & 0 & yu \\ 0 & 0 & u & 0 & zu \\ 0 & 0 & 0 & u & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0,0,0,0), (0,0,0,1), (0,0,0,2),$ $(0,0,1,4), (0,0,1,2), (0,1,1,2), (0,1,2,3), (0,1,1,4), (1,1,1,2), (1,4,2,3), (1,4,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & xu & yu & 2 \\ 0 & 1 & zu & 2 & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0,0,0,0), (0,0,1,0), (0,1,0,4),$ $(0,1,1,4), (1,0,1,0), (1,1,1,4), (1,1,2,4)\}$;
- $\begin{bmatrix} 1 & 0 & 1 & xu+2 & yu+2 \\ 0 & 1 & zu+2 & wu+1 & su+3 \end{bmatrix}$ with $(x, y, z, w, s) \in \{(0,0,0,0,0),$ $(0,0,1,4,3), (0,0,2,3,1), (1,4,0,4,2), (1,4,1,3,0), (1,4,2,2,3), (2,3,0,3,4),$ $(2,3,2,1,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & 2 \\ 0 & 1 & 0 & 2 & yu \\ 0 & 0 & u & 0 & 0 \end{bmatrix}$ with $(x, y) \in \{(0,0), (1,4)\}$;
- $\begin{bmatrix} 1 & 0 & 1 & xu+2 & yu+2 \\ 0 & 1 & 2 & zu+1 & wu+3 \\ 0 & 0 & u & 4u & 3u \end{bmatrix}$ with $(x, y, z, w) \in \{(0,0,0,0), (1,4,4,2)\}$.

A self-orthogonal code of the type $\{k_1, k_2\}$ and length n over $\mathbb{F}_q[u]/ < u^2 >$ is self-dual if and only if

$$2k_1 + k_2 = n.$$

There are precisely

- 2 inequivalent self-dual codes of length 3 over $\mathbb{F}_5[u]/ < u^2 >$.
- 5 inequivalent self-dual codes of length 4 over $\mathbb{F}_5[u]/ < u^2 >$.
- 8 inequivalent self-dual codes of length 5 over $\mathbb{F}_5[u]/ < u^2 >$.

Total number of LCD and inequivalent LCD codes of a given rank and length 4 over $\mathbb{F}_2[u]/ < u^2 >$

Rank k	Total number of LCD codes of length 4 and rank k over $\mathbb{F}_2[u]/ < u^2 >$	Number of inequivalent LCD codes of length 4 and rank k over $\mathbb{F}_2[u]/ < u^2 >$
1	64	8
2	320	24
3	64	8
4	1	1

There are precisely 41 inequivalent non-zero LCD codes of length 4 over $\mathbb{F}_2[u]/<u^2>$, whose generator matrices are as listed below:

- $\begin{bmatrix} 1 & xu & yu & zu \end{bmatrix}$ with $(x,y,z) \in \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}$;
- $\begin{bmatrix} 1 & 1+xu & 1+yu & zu \end{bmatrix}$ with $(x,y,z) \in \{(1,1,1),(1,1,0),(0,0,1),(0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & xu & yu \\ 0 & 1 & zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(1,0,0,1),(1,1,0,1),(1,1,1,1),$ $(0,1,0,1),(0,0,0,1),(0,0,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu \\ 0 & 1 & zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(1,0,1,1),(1,1,1,1),$ $(0,1,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu \\ 0 & 1 & zu & 1+wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0,),(1,1,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu \\ 0 & 1 & 1+zu & 1+wu \end{bmatrix}$ with $(x,y,z,w) \in \{(1,1,0,0),(1,1,1,0),(1,1,1,1),$ $(0,1,0,1),(0,1,1,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & yu \\ 0 & 1 & 1+zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,1,0,0),(0,1,1,1),(0,0,0,0),$ $(0,0,1,0),(0,0,1,1)\}$;
- I_4, $\begin{bmatrix} 1 & 0 & u & u \\ 0 & 1 & 1 & 1 \end{bmatrix}$;

- $\begin{bmatrix} 1 & 0 & 0 & xu \\ 0 & 1 & 0 & yu \\ 0 & 0 & 1 & zu \end{bmatrix}$ with $(x, y, z) \in \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu \\ 0 & 1 & 0 & 1+yu \\ 0 & 0 & 1 & zu \end{bmatrix}$ with $(x, y, z) \in \{(1,1,1), (1,1,0), (0,0,0), (0,0,1)\}$.

Total number of LCD and inequivalent LCD codes of a given rank and length 5 over $\mathbb{F}_2[u]/<u^2>$

Rank k	Total number of LCD codes of length 5 and rank k over $\mathbb{F}_2[u]/<u^2>$	Number of inequivalent LCD codes of length 5 and rank k over $\mathbb{F}_2[u]/<u^2>$
1	256	14
2	5120	81
3	5120	81
4	256	14
5	1	1

There are precisely 191 inequivalent non-zero LCD codes of length 5 over $\mathbb{F}_2[u]/<u^2>$, whose generator matrices are as listed below:

- $\begin{bmatrix}1 & xu & yu & zu & wu\end{bmatrix}$ with $(x,y,z,w) \in \{(0,1,0,1),(0,0,1,0),(1,0,1,1),(0,0,0,0),(1,1,1,1)\}$;
- $\begin{bmatrix}1 & 1+xu & 1+yu & zu & wu\end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(1,0,0,1),(0,0,0,1),(1,0,1,1),(1,1,0,0),(0,0,1,1)\}$;
- $\begin{bmatrix}1 & 1+xu & 1+yu & 1+zu & 1+wu\end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,1,1),(1,1,1,1),(0,0,0,0)\}$;
- $\begin{bmatrix}1 & 0 & xu & yu & zu\\ 0 & 1 & wu & su & tu\end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,0,0,1,1),(1,0,0,0,1,1),(0,0,0,0,1,0),(1,1,1,0,1,0),(0,0,0,0,0,0),(1,1,1,0,0,0),(1,1,0,1,1,1),(1,1,0,1,0,1),(1,1,0,1,1,0),(1,1,0,0,0,0),(0,0,1,0,0,1),(1,1,1,1,1,1),(0,0,1,0,1,0)\}$;
- $\begin{bmatrix}1 & 0 & xu & yu & zu\\ 0 & 1 & 1+wu & 1+su & tu\end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,0,0,1,0,0),(1,1,1,0,1,0),(1,1,0,1,1,1),(1,0,0,1,1,0),(0,0,1,1,1,0),(1,0,0,0,0,0),(0,0,1,0,0,0),(0,0,0,0,1,1),(0,0,0,0,0,0),(1,1,1,0,0,0),(0,1,1,0,1,0)\}$;
- $\begin{bmatrix}1 & 0 & 1+xu & yu & zu\\ 0 & 1 & 1+wu & 1+su & 1+tu\end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,1,0,0,0),(1,1,0,1,0,1),(0,0,1,0,1,1),(0,0,1,1,0,1)\}$;

- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu & zu \\ 0 & 1 & 1+wu & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,0,0,1,0),$ $(1,0,1,0,1,0), (0,1,1,0,1,1), (0,0,1,0,0,0), (1,1,0,0,0,0), (0,0,1,1,1,0),$ $(1,0,0,1,0,1), (0,1,1,1,1,1), (1,1,0,1,1,0), (1,0,1,0,0,0), (0,1,0,1,0,0),$ $(1,0,0,1,0,0), (0,0,0,0,0,1), (1,0,0,0,0,1), (0,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu & zu \\ 0 & 1 & 1+wu & 1+su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,1,0,1,0),$ $(0,1,1,0,1,1), (0,1,1,1,0,1), (1,1,0,1,1,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & yu & zu \\ 0 & 1 & 1+wu & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,0,0,0,1),$ $(0,0,1,0,1,1), (1,1,1,1,1,1), (0,1,1,1,0,0), (0,1,1,0,1,1), (1,1,0,0,1,1),$ $(1,0,1,0,0,0), (1,0,0,0,0,0), (1,0,1,1,0,1), (1,0,1,1,0,0), (0,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu & zu \\ 0 & 1 & wu & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,1,1,0,1),$ $(1,1,0,1,1,0), (0,0,1,1,1,0), (0,1,1,0,0,1), (0,0,1,0,0,1), (0,0,1,1,0,1),$ $(1,1,1,1,0,0), (1,1,1,1,1,1), (0,0,1,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & yu & zu \\ 0 & 1 & 1+wu & 1+su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,1,1,1,0),$ $(0,0,1,0,1,0), (1,0,1,1,0,1), (1,0,0,0,0,1), (0,1,1,0,1,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu & 1+zu \\ 0 & 1 & 1+wu & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,1,1,0,1),$ $(1,1,1,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,0)\}$;

- $\begin{bmatrix} 1 & 0 & 1+xu & 1+yu & 1+zu \\ 0 & 1 & 1+wu & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,1,1,0,1),$ $(0,0,0,1,0,0),(0,0,1,0,0,0),(1,0,1,1,0,0),(1,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,0,1,1,1),$ $(1,1,0,0,1,1),(1,0,1,1,0,1),(0,0,0,1,1,0),(0,0,0,0,1,0),(0,1,1,1,0,1),$ $(0,1,0,1,0,0),(1,0,1,1,0,0),(0,0,1,1,0,0),(1,1,1,1,1,1),(1,0,1,0,0,1),$ $(1,0,1,0,1,0),(0,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & 1+yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,1,0,0,0),$ $(0,0,0,0,1,0),(1,0,1,1,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & 1+yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,0,1,0,0,1),$ $(1,1,1,1,1,1),(0,1,0,1,0,1),(0,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,0,0,0,0,1),$ $(0,0,1,0,1,0),(0,1,0,1,0,1),(1,0,1,1,0,0),(0,1,1,1,1,0),(1,1,1,1,0,0)\}$;

- $\begin{bmatrix} 1 & 0 & 0 & xu & yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,1,0,0,0), (1,0,0,1,1,0), (1,0,0,1,0,0), (0,0,0,1,0,1)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & 1+yu \\ 0 & 1 & 0 & zu & 1+wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,1,1,1,0), (1,1,0,1,0,0), (1,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,1,0,0,1), (1,0,1,0,1,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & 1+yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,1,1,1,1,0), (0,0,0,1,0,1), (0,0,0,0,0,1), (1,1,0,0,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & yu \\ 0 & 1 & 0 & 1+zu & wu \\ 0 & 0 & 1 & 1+su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,0,0,0,0,0), (1,0,0,1,0,0), (0,1,1,0,1,0), (1,1,0,1,0,0), (1,1,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & 1+yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,0,1,1,1,0), (0,1,0,1,0,1), (1,1,1,0,1,1)\}$;

- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & yu \\ 0 & 1 & 0 & 1+zu & wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,1,0,0,0),(0,1,1,1,0,1),$ $(0,1,0,0,0,0),(0,0,1,1,1,1),(0,1,0,0,0,1),(0,0,0,0,0,1)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & 1+yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,1,0,1,1),$ $(1,1,1,1,1,1)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & 1+su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,0,0,1,0),(0,1,1,0,1,1),$ $(1,1,1,0,1,1),(1,0,1,0,0,0),(1,1,0,0,1,1),(0,0,1,1,1,0),(1,1,0,1,1,1),$ $(1,1,1,1,1,0),(0,0,1,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,1,1,1,0),$ $(0,0,0,0,1,1),(0,1,1,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu & 1+yu \\ 0 & 1 & 0 & 1+zu & 1+wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,0,1,0,0),$ $(0,0,0,0,0,0)\}$;

- $\begin{bmatrix} 1 & 0 & 0 & xu & 1+yu \\ 0 & 1 & 0 & zu & 1+wu \\ 0 & 0 & 1 & 1+su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,0,1,0,0,1),$ $(0,0,1,1,0,1),(0,0,0,1,0,0),(1,0,1,1,0,1),(0,1,0,1,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+u & 1 \\ 0 & 1 & 0 & u & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 1+u & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & 1+yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & su & 1+tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(0,0,0,1,0,0),$ $(1,1,0,1,1,1),(0,0,1,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu & 1+yu \\ 0 & 1 & 0 & zu & wu \\ 0 & 0 & 1 & su & tu \end{bmatrix}$ with $(x,y,z,w,s,t) \in \{(1,1,0,1,1,1),$ $(0,0,0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 0 & xu \\ 0 & 1 & 0 & 0 & yu \\ 0 & 0 & 1 & 0 & zu \\ 0 & 0 & 0 & 1 & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,1,1,1),(0,0,0,0),(1,1,1,1),$ $(0,1,1,0),(0,0,1,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 0 & 1+xu \\ 0 & 1 & 0 & 0 & 1+yu \\ 0 & 0 & 1 & 0 & 1+zu \\ 0 & 0 & 0 & 1 & 1+wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,1,1,0),(0,0,1,0),(0,0,0,0)\}$;

- $\begin{bmatrix} 1 & 0 & 0 & 0 & xu \\ 0 & 1 & 0 & 0 & 1+yu \\ 0 & 0 & 1 & 0 & 1+zu \\ 0 & 0 & 0 & 1 & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,1,1,0),(0,0,0,0)\}$;
- I_5, $\begin{bmatrix} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1+u \\ 0 & 0 & 0 & 1 & 1+u \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & u \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & u \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$,
$\begin{bmatrix} 1 & 0 & 0 & 0 & 1+u \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & u \\ 0 & 0 & 0 & 1 & u \end{bmatrix}$.

Total number of LCD and inequivalent LCD codes of a given rank and length 4 over $\mathbb{F}_3[u]/<u^2>$

Rank k	Total number of LCD codes of length 4 and rank k over $\mathbb{F}_3[u]/<u^2>$	Number of inequivalent LCD codes of length 4 and rank k over $\mathbb{F}_3[u]/<u^2>$
1	648	15
2	7290	63
3	648	15
4	1	1

There are precisely 94 inequivalent non-zero LCD codes of length 4 over $\mathbb{F}_3[u]/<u^2>$, whose generator matrices are as listed below:

- $\begin{bmatrix}1 & xu & yu & zu\end{bmatrix}$ with $(x, y, z) \in \{(2, 0, 0), (0, 1, 1), (2, 1, 1), (0, 0, 0)\}$;
- $\begin{bmatrix}1 & 1+xu & yu & zu\end{bmatrix}$ with $(x, y, z) \in \{(0, 0, 2), (2, 1, 1), (0, 1, 1)\}$;
- $\begin{bmatrix}1 & 2+xu & yu & zu\end{bmatrix}$ with $(x, y, z) \in \{(0, 0, 0), (2, 2, 0), (1, 0, 0)\}$;
- $\begin{bmatrix}1 & 1+xu & 1+yu & 2+zu\end{bmatrix}$ with $(x, y, z) \in \{(1, 1, 1), (2, 2, 1), (0, 0, 0), (2, 0, 1)\}$;
- $\begin{bmatrix}1 & 1+u & 2+2u & 2+2u\end{bmatrix}$;
- $\begin{bmatrix}1 & 0 & 2+xu & 1+yu \\ 0 & 1 & 1+zu & wu\end{bmatrix}$ with $(x, y, z, w) \in \{(0, 1, 0, 2), (0, 2, 0, 1), (2, 1, 2, 0)\}$;
- $\begin{bmatrix}1 & 0 & 1+xu & 2+yu \\ 0 & 1 & 1+zu & wu\end{bmatrix}$ with $(x, y, z, w) \in \{(0, 1, 0, 2), (2, 1, 2, 0), (2, 1, 1, 0), (2, 0, 1, 0), (1, 0, 2, 1), (0, 0, 0, 0), (1, 1, 0, 0)\}$;
- $\begin{bmatrix}1 & 0 & 1+xu & 2+yu \\ 0 & 1 & 1+zu & 2+wu\end{bmatrix}$ with $(x, y, z, w) \in \{(0, 1, 0, 1), (1, 0, 1, 0), (1, 2, 0, 0), (1, 1, 0, 0), (1, 0, 0, 2), (0, 2, 2, 2)\}$;
- $\begin{bmatrix}1 & 0 & 1+xu & yu \\ 0 & 1 & zu & wu\end{bmatrix}$ with $(x, y, z, w) \in \{(2, 1, 0, 0), (1, 0, 0, 0), (0, 1, 2, 0), (1, 1, 1, 0), (0, 1, 0, 1), (1, 2, 0, 1), (1, 2, 1, 1), (2, 1, 2, 1), (2, 0, 0, 1), (2, 0, 1, 1)$ $(0, 0, 0, 2), (0, 0, 2, 0)\}$;

- $\begin{bmatrix} 1 & 0 & 1+xu & yu \\ 0 & 1 & zu & 1+wu \end{bmatrix}$ with $(x,y,z,w) \in \{(1,1,1,0),(1,1,0,0),(2,0,2,0),$ $(1,0,0,0),(2,2,2,2),(1,1,1,2),(1,1,0,2),(1,0,2,0),(1,2,1,1),(0,2,0,0),$ $(2,0,0,2),(0,2,1,0),(0,0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 2+xu & yu \\ 0 & 1 & 1+zu & 1+wu \end{bmatrix}$ with $(x,y,z,w) \in \{(2,1,2,0),(2,2,2,0),(2,1,1,0),$ $(2,2,1,0),(2,2,0,2),(1,2,1,2),(2,1,1,2),(0,0,1,0),(0,0,2,0)\}$;
- $\begin{bmatrix} 1 & 0 & xu & yu \\ 0 & 1 & 1+zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(0,0,0,0),(2,2,0,0),(2,2,0,1),$ $(0,0,0,1)\}$;
- $\begin{bmatrix} 1 & 0 & xu & yu \\ 0 & 1 & zu & wu \end{bmatrix}$ with $(x,y,z,w) \in \{(2,2,0,0),(0,0,0,0),(1,2,1,1),$ $(1,2,1,2),(2,2,2,0),(1,0,2,0),(0,2,0,0),(0,2,2,0)\}$;
- $\begin{bmatrix} 1 & 0 & 1 & 2+u \\ 0 & 1 & 2+u & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1+u & u \\ 0 & 1 & u & 1+u \end{bmatrix};$
- $\begin{bmatrix} 1 & 0 & 0 & xu \\ 0 & 1 & 0 & yu \\ 0 & 0 & 1 & zu \end{bmatrix}$ with $(x,y,z) \in \{(1,1,0),(2,0,0),(1,1,1),(0,0,0)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & xu \\ 0 & 1 & 0 & 2+yu \\ 0 & 0 & 1 & zu \end{bmatrix}$ with $(x,y,z) \in \{(1,2,1),(0,2,0),(2,0,1),(0,0,2)\}$;
- $\begin{bmatrix} 1 & 0 & 0 & 1+xu \\ 0 & 1 & 0 & 2+yu \\ 0 & 0 & 1 & 1+zu \end{bmatrix}$ with $(x,y,z) \in \{2,1,0),(0,0,1),(0,0,2)\}$;

- I_4, $\begin{bmatrix}1 & 0 & 0 & 2+u\\ 0 & 1 & 0 & 2+2u\\ 0 & 0 & 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 0 & 2+2u\\ 0 & 1 & 0 & 2u\\ 0 & 0 & 1 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$.

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Thank you...